chapter 12: Comparison Method and Stability of EPCA.

Goal:

Systems of nonlinear differential equations with piecewise constant arguments (EPCAs) $\Rightarrow$ { comparison principle. $\Downarrow$ stability properties.

Results: piecewise (constant) argumenes

un stable $\Longrightarrow$ stable.

Introduction: nonlinear EPCA has the form

$$\dot{x} = f(t, x(t), x(\gamma(t))).$$

where the argumene γ is a piecewise constant function defined on intervals with a certain length. For example. $[\cdot]$ greatest integer function

$$\gamma(t) = [t],\ [t-n],\ t-n[t],\ [t+1].\ \forall t, n$$

problem Formulation.

Let $\{t_k\}_{k\geq 0}^{\infty}$ and $\{\xi_k\}_{k\geq 0}^{\infty}$ be sequences of nonnegative real numbers such that $\lim_{k\to\infty} t_k = \infty$, $t_{k-1} \leq \xi_k \leq t_k$, $\xi_0 = t_0$.

Consider the EPCA of the form.

$$\dot{x}(t) = f(t, x(t), \lambda_{p(t)}(x(\gamma(t)))). \quad (12.2a).$$

$p(t) = k$, $\gamma(t) = \xi_k$. $t \in [t_k, t_{k+1}]$ (piece wise constant. functions)

$$x(t_0) = x_0. \quad (12.2b).$$

For (12.2), we state the soultion of IVP.

Definition 12.1. $x: (\alpha, \beta) \to \mathbb{R}$ is said to be a solution of (12.2) if the following conditions hold:

(i) $x(t)$ is continuous for all $t \in (\alpha, \beta)$;

(ii) $\dot{x}(t)$ exists and continnous for all $t \in (\alpha, \beta) \setminus \{t = \xi_k\}$

(iii) $\dot{x}(t)$ satisfies the EPCA in (12.2a);

(iv) $x(t)$ satisfies the initial condition in (12.2b) at $t = t_0$.

We can rewrite (12.2) as follows.

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda_k(x_{s_k})), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \dots & (12.3a) \\ x(t_0) = x_0 & (12.3b) \end{cases}$$

We employ the theory of ODE to show the solution of (12.3).

For $k=0$, $t \in [t_0, t_1)$, the IVP

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda_0(x_{s_0})), \\ x(t_0) = x_0, \quad s_0 = t_0 \end{cases}$$

has a unique solution, say $x_0(t)$, and $x_0(t_1^-) = \lim_{t \to t_1^-} x_0(t)$.

Similarly, for $k=1$ and $t \in [t_1, t_2)$, we have the IVP

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda_1(x_{s_1})), \\ x(t_1) = x(t_1^-), \end{cases}$$

which has a unique soultion, say $x_1(t)$.

By induction, we have

$$x(t) = \begin{cases} x_0 & t = t_0 \\ x_0(t, t_0, x_0) & t \in [t_0, t_1) \\ x_1(t, t_1, x_1) & t \in [t_1, t_2) \quad x_1 = x_0(t_0^-, t_0, x_0). \\ \vdots & \\ x_k(t, t_k, x_k) & t \in [t_k, t_{k+1}) \quad , \quad x_k = x_{k-1}(t_k^-, t_{k-1}, x_{k-1}). \end{cases}$$

To further analyze the comparison principle, we define auxiliary scalar system

$$\begin{cases} \dot{u}(t) = g(t, u(t), \sigma_k(u_{s_k})) & (12.4a) \\ u(t_0) = u_0 & (12.4b) \end{cases}$$

Definition 12.2. (Lyapunov Functional).

If $V \in \varphi([t_k, t_{k+1}) \times \mathbb{R}^d, \mathbb{R}_+)$, the upper right-hand derivate of V is defined by

$$D^+V(t, x, y) = \limsup_{h \to 0^+} \frac{1}{h}\left[V(t, x+h f(t, x, \lambda_k(y))) - V(t, x) \right].$$

moreover, if $V \in \varphi^1([t_k, t_{k+1}) \times \mathbb{R}^n, \mathbb{R}_+)$, then

$$D^+V(t, x, y) = \frac{\partial V(t,x)}{\partial t} + \nabla V(t, x) \cdot f(t, x, \lambda_k(y)).$$

12.3. Comparison Method.

Theorem 12.1. Assume that the following conditions hold:

(i) for $k=0,1,2,\ldots$, $V \in \mathcal{V}([t_k, t_{k+1}) \times R^n; R_+)$, $V(t,x)$ is locally Lipschitz in x and.

$$D^+V(t,x,V_{\xi_k}) \le g(t, V(t,x), \sigma_k(V_{\xi_k})). \quad t \in (t_k, t_{k+1})$$

where $V_{\xi_k} = V(\xi_k, x(\xi_k))$; and.

(ii) the maximal solution $\theta(t;t_0,u_0)$ of auxiliary scalar EPCA-(12.4) exists on $[t_0, \infty)$.

Then $V(t_0,x_0) \le u_0$ implies $V(t,x(t)) \le \theta(t,t_0,u_0)$, $t \ge t_0$

Proof: Define $m(t) = V(t,x(t))$.

$$D^+m(t) \le g(t, m(t), \sigma_k(m_{\xi_k})) \quad t \in (t_k, t_{k+1}).$$

Thus. for $t \in [t_0, t_1]$, by classical comparison principle [6].

$$m(t) \le \theta_0(t; t_0, u_0) \quad t \in [t_0, t_1],$$

where $\theta_0(t;t_0,u_0)$ is the maximal solution of auxiliary scalar.

IVP
$$\begin{cases} \dot{u}(t) = g(t, u(t), \sigma_0(u_{\xi_0})). \\ u(t_0) = u_0 \end{cases}$$

By induction, we define

$$u(t) = \begin{cases} u_0 & t = t_0 \\ \theta_0 & t \in [t_0, t_1] \\ \theta_1 & t \in (t_1, t_2] \\ \vdots & \vdots \\ \theta_k & t \in (t_k, t_{k+1}] \\ \vdots & \vdots \end{cases}$$

Then. for $t \ge t_0$, we get.

$$m(t) \le u(t). \Rightarrow m(t) \le \theta(t; t_0, u_0)$$

With the above preparations. we address some special cases of EPCA and EPCAG.

Corollary 12.1. Suppose that the condition in Theorem 12.1 hold, Let $k = 0, 1, 2, \ldots$ and $t \in [t_k, t_{k+1}]$. If we choose.

(i) $g(t, u, \sigma_k(u_{\xi_k})) = \beta_k u_{\xi_k}$, with β_k being a constant for all k, then

(1) $\xi_k = t_k$, we have

$$V(t, x(t)) \le \begin{cases} [1+\beta(t-t_0)] V(t_0, x_0) & k>0,\ t \in [t_0, t_1) \\ [1+\beta_k(t-t_k)] \prod_{j=1}^{k} [1+\beta_{j-1}(t_j - t_{j-1})] V(t_0, x_0). & k \in \mathbb{N},\ t \in [t_k, t_{k+1}) \end{cases}$$

where $t_k < t_{k+1}$ if $\beta_k > 0$ and $t_{k+1} < t_k - \frac{1}{\beta_k}$ if $\beta_k < 0$;

(2) $t_{k-1} \le \xi_k \le t_k$, ~~where~~ we have

$$V(t, x(t)) = V_0(t, x(t)) \le [1+\beta_0(t-t_0)] V_0(t_0, x_0).$$

$\forall t \in [t_0, t_1)$. such that $t_1 - t_0 < -\frac{1}{\beta_0}$ and

$$V(t, x(t)) = V_k(t, x(t)) \le V_{k-1}(t_k, x(t_k)) + \beta_k (t - t_k) V_{k-1}(\xi, x(\xi_k))$$

for any $t \in [t_k, t_{k+1})$, $t_{k+1} - t_k < -\frac{C_k}{\beta_k C_{\xi_k}}$ where $C_k = V_{k-1}(t_{k-1}, x(t_{k-1}))$ / $C_{\xi_k} = V_{k-1}(\xi_k, x(\xi_k))$

(ii) $g(t, u, \sigma(u_{\xi_k})) = \alpha u(t) + \beta_k u_{\xi_k}$

(iii) $g(t, u, \sigma(u_{\xi_k})) = \alpha u(t) + h(t, u, \sigma_k(u_{\xi_k}))$.

proof. We only prove (i).

(i)(1) For $t \in [t_k, t_{k+1}]$, since $u_{\xi_k} = u_{t_k}$, the solution of the differential equation $\dot{u}(t) = \beta_k u_{\xi_k}$ is given by

$$u(t) = [1+\beta_k (t-t_k)] u_k.$$

In particular, for $k>0$, $t \in [t_0, t_1]$. we have

$$u(t) = [1+\beta_0(t-t_0)] u_0$$

for $k=1$. $t \in [t_1, t_2]$. we have

$$u(t) = [1+\beta_1(t-t_1)][1+\beta_0(t_1-t_0)] u_0.$$

Thus. by induction, we reach

$$u(t) = \begin{cases} [1+\beta_0(t-t_0)]u_0 & , k=0, t\in[t_0, t_1] \\ [1+\beta_k(t-t_k)] \prod_{j=1}^{k} [1+\beta_{j-1}(t_j-t_{j-1})] u_0 & k\in N, t\in(t_k, t_{k+1}] \end{cases}$$

By comparison result developed in Th 2.1, we complete the proof.

(i)(2). For any k, $t\in[t_k, t_{k+1})$

$$u(t) = u(t_k) + \beta_k(t-t_k)u(\xi_k).$$

For $k=0$, $\xi_0=t_0$

$$u(t) = [1+\beta_0(t-t_0)]u_0 =: u_0(t). \quad (\beta_0<0, t<t_0-\tfrac{1}{\beta_0})$$

For $k=1$ ξ_1, $u(t) = u_0(t_1) + \beta(t-t_1)u_0(\xi_1) =: u_1(t)$

By induction, we reach.

$$u(t) = u_k(t) = u_{k-1}(t_k) + \beta_k(t-t_k)u_{k-1}(\xi_k).$$

This implies the desired estimates.

12.4. Stability Analysis.

Having established the comparison results in Theorem 12.1. we provide some stability notions for the nonlinear EPCA.

Theorem 12.2. In addition to the conditions in Theorem 12.1, assume further that there exist class-K function a and b such that

$$b(\|x\|) \le V(t,x) \le a(\|x\|) \qquad (12.6)$$

Then stability of $u\equiv0$ in (12.4) $\Rightarrow$ stability of $x\equiv0$ in (12.3).

proof. Fix $t_0\in R_+$. $\varepsilon>0$. Suppose $u\equiv0$ is stable.

$\forall\, b(\varepsilon) > 0.\ \exists\, \delta_1 = \delta_1(t_0, \varepsilon)$. s.t.

$0 \le u_0 \le \delta_1 \Rightarrow u(t; t_0, u_0) \le b(\varepsilon)\ \forall t \ge t_0.$

choose $\delta_2 = \delta_2(\varepsilon)$ such that $a(\delta_2) \underset{<\delta_1}{<} b(\varepsilon)$. Define $\delta = \min\{\delta_1, \delta_2\}$

Next. we prove $x \equiv 0$ is stable; that is, if $\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon,\ \forall t \ge t_0$

suppose not. , there exist some $t^* \ge t_0$ such that. ($\|x_0\| < \delta$).

$$\|x(t^*)\| \ge \varepsilon$$

Let $u_0 = a(\|x_0\|) \le a(\delta) \doteq a(\delta_2) < \delta_1$ and define $m(t) = V(t, x(t))$. $t_0 < t < \tilde{t}$

Then. by Theorem 12.1. we have

$$V(t, x(t)) \le \Theta(t; t_0, a(\|x_0\|))$$

by stability of u

$$\Rightarrow b(\varepsilon) \le b(\|x^*\|) \le V(\tilde{t}, \tilde{x}(t)) \le \Theta(\tilde{t}, t_0, \underbrace{a(\|x_0\|)}_{\le \delta_1}) < b(\varepsilon)$$

which is a contradiction. This shows $x \equiv 0$ is stable.

If, moreover, δ is independent of t_0. then $x \equiv 0$ is uniformly stable

Now. we use asymptotic stability of $u \equiv 0$ to show as. stability of $x \equiv 0$.

$\Downarrow$

$\forall\, b(\varepsilon) > 0.\ \exists\, \delta_0^* = \delta^*(t_0) > 0.\ T = T(t_0, \varepsilon)$ such that

$0 \le u_0 \le \delta^*$ implies $u(t, t_0, u_0) < b(\varepsilon)$, $\forall t \ge t_0 + T$.

choose a $\tilde{\delta}$ such that $a(\tilde{\delta}) < \delta_0^*$. $\rho = \min\{\delta_0^*, \tilde{\delta}\}$, $\|x_0\| < \rho$

$$\Rightarrow \quad b(\|x(t)\|) \le V(t, x(t)) \le \Theta(t, t_0, a(\|x_0\|)) < b(\varepsilon).$$

$$\Rightarrow \|x(t)\| \le b^{-1}(b(\varepsilon)). \quad \Rightarrow x \equiv 0 \text{ is asymptotically stable}$$

corollary 12.2. $\quad g(t, u(t), \sigma_k(u_{\xi_k})) = \beta_k u_{\xi_k}$.

(i) In the case $\xi_k \ge t_k$.

(1) if $\beta_k \ge 0$ for any k and the infinite series.

$$\sum_{j=1}^{\infty} \beta_{j-1}(t_j - t_{j-1}). \qquad (18.a)$$

converges. then $x=0$ is uniformly stable.

proof: (i) (1) $\begin{cases} u'(t) = \beta_k u_{\xi_k}, & t \in [t_k, t_{k+1}] \\ u(t_0) = u_0 \end{cases}$

is given by

$$u(t) = (1+\beta_k(t-t_k)) \prod_{j=1}^{k} [1+\beta_j(t_j - t_{j-1})] u_0.$$

By (12.8a), the product $\prod_{j=1}^{\infty} [1+\beta_j(t_j - t_{j-1})]$ converges

$\prod_{a=1}^{\infty} (1+x_a)$ $\sum_{a=1}^{\infty} x_a < \infty$. $\prod_{a=1}^{\infty} (1+x_a) = e^{\ln \prod_{a=1}^{\infty} (1+x_a)} = e^{\sum_{a=1}^{\infty} \ln(1+x_a)} \le e^{\sum_{a=1}^{\infty} x_a} < \infty.$

So that, defining $M = \prod_{j=1}^{\infty} [1+\beta_j(t_j - t_{j-1})]$

$\Rightarrow$ $u(t, t_0, u_0) = M u_0 < M\delta$, for some δ such that $u_0 < \delta$.

$\Rightarrow$ $u=0$ is uniformly stable.

Corollary 12.3 $g(t, u(t), \sigma_k(u_{\xi_k})) = \alpha u(t) + \beta_k u_{\xi_k}$

Cor: 12.4 $g(t, u(t), \sigma_k(u_{\xi_k})) = -w(u) + \beta_k u_{\xi_k}$. $w \in \mathcal{K}$.

Cor: 12.5 $g(t, u(t), \sigma_k(u_{\xi_k})) = \lambda u(t) + h(t, u(t), \sigma_k(u_{\xi_k}))$.

Conclusion.

comparison method + stability analysis

$\Downarrow$

stabilize condition..

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